(Hidden) Assumptions of Simple ODE Models

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Meaningful Modeling of Epidemiological Data
(MMED) clinic, ICI3D Program, AIMS - South Arica

Model terminology

- Deterministic
- Stochastic

- Continuous time
- Discrete time

- Compartmental models
- Network models
- Individual-based models

Differential equations (ODE's)

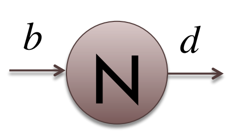
- Equations describe the change in state variables through time
  - → *deterministic progression from a set of initial conditions*

- Good for:
  - understanding periodicity in long time series for large populations
  - understanding effects of vaccination and birth rates on persistence and periodicity

Differential equations (ODE's)

- Continuous treatment of individuals; appropriate for:
  - average system behavior
  - population proportions
  - population densities

- Continuous treatment of time



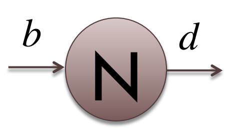
$$\frac{dN}{dt} = bN - dN$$

Differential equations (ODE's)

- Assumptions
  - large (infinite) populations
  - well-mixed contacts
  - homogenous individuals
  - exponential waiting times (memory-less)

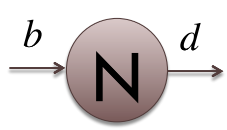
Implications of continuous time

- Continuous treatment of time



$$\frac{dN}{dt} = bN - dN$$

- Treatment of time as discrete steps



$$\frac{\Delta N}{\Delta t} = bN - dN$$

Discrete time

$$\frac{\Delta N}{\Delta t} = bN - dN$$

- N is the population size or density
- t is time
- Δ denotes "change in"

Discrete time

$$\frac{\Delta N}{\Delta t} = bN - dN$$

- b is the ***per capita*** birth rate
- b*N is the total birth rate
- d is the ***per capita*** death rate
- d*N is the total death rate

Discrete time

$$\frac{\Delta N}{\Delta t} = bN - dN$$

- This equation can be multiplied by Δt to get:

$$\Delta N = (bN - dN)\Delta t$$

- ***What are the units of each side of the new equation?***

Discrete time

$$\Delta N = (bN - dN)\Delta t$$

- If we define r = b – d (often called the "intrinsic population growth rate"), this equation can be rewritten as:

$$\Delta N = rN\Delta t$$

$$\Delta N = r N \Delta t$$

- If we know the state of the population, N_t, at some time t, then we can calculate the state of the population at time t + Δt as:

$$N_{t+\Delta t} = N_t + \Delta N$$

- In this case:

$$N_{t+\Delta t} = N_t + r N \Delta t$$

- So, if r>0, N gets bigger with time; if r<0, N gets smaller with time; and if r=0, then

$$N_{t+\Delta t} = N_t$$

Example

- Say we have a population with an intrinsic population growth rate of r = 24 day^{-1}
- If we start at time t = 0 with N_0 = 1 individual, and our population reproduces at this rate every day (Δt = 1),after 1 day we would expect to have a population size of

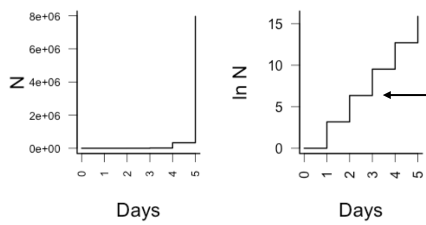
$$N_{t+\Delta t} = N_t + r N \Delta t$$
$$N_1 = N_0 + 24 * 1 * 1$$
$$N_1 = 25$$

Example

- After two days, we would have a population size of

$$N_2 = N_1 + r N_1 \Delta t$$
$$N_2 = 25 + 24 * 25 * 1$$
$$N_2 = 625$$

- Since r>0, the population grows exponentially:



$$N_2 = 625 \approx e^{6.44}$$

Example 2

- Now, say we have a population with an intrinsic population growth rate of r = 24 day^{-1} but our population reproduces every hour, instead of every day (so $\Delta t = 1/24$ because our unit of time is still days)
- If we start at time t = 0 with $N_0 = 1$ individual, after 1 hour we would expect to have a population size of

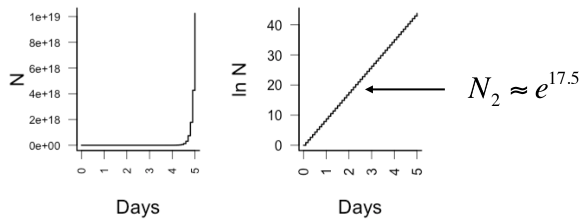
$$N_{1/24} = N_0 + 24 * 1 * 1/24$$
$$N_{1/24} = 2$$

Example 2

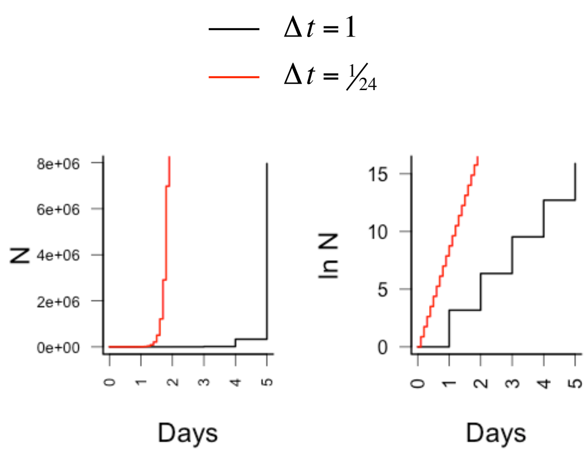
- After two hours, we would have a population size of

$$N_{2/24} = N_{1/24} + r\,N_{1/24}\,\Delta t$$
$$N_{2/24} = 2 + 24 * 2 * 1/24$$
$$N_{2/24} = 4$$

- The population still grows exponentially:



Examples 1 & 2



Example 3

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t}$$

Example 3

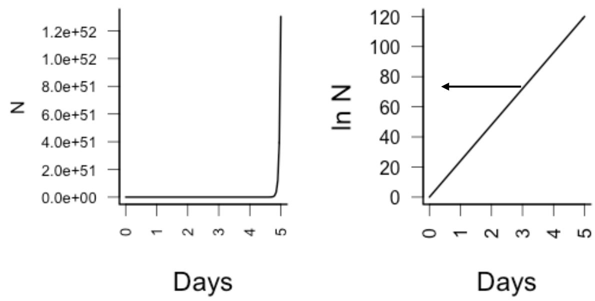
$$\frac{dN}{dt} = bN - dN = rN$$

- This is known as an ordinary differential equation model

Example 3

- Using calculus, we can show that

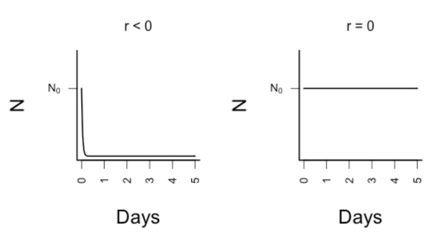
$$N_t = N_0 e^{rt}$$



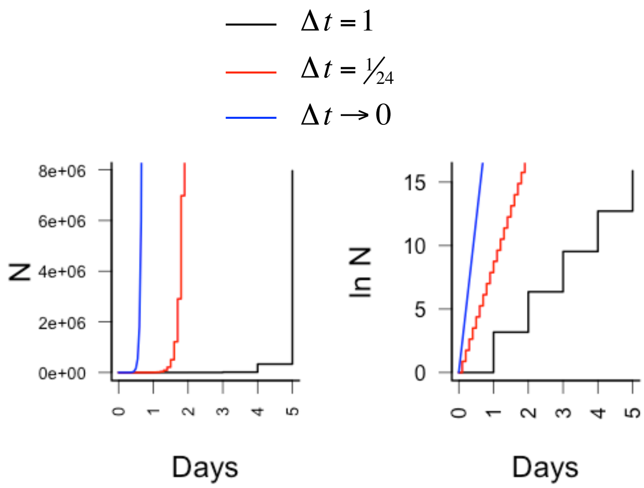
$$N_2 = e^{48}$$

$$N_t = N_0 e^{rt}$$

- As before, the population increases exponentially when r > 0
- When r < 0, the population experiences exponential decline, and when r = 0, the population remains constant



Implications of continuous time



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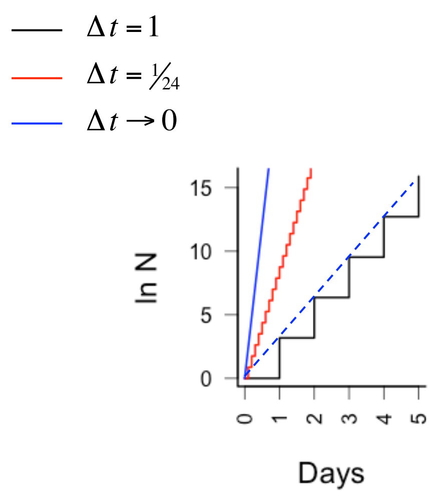

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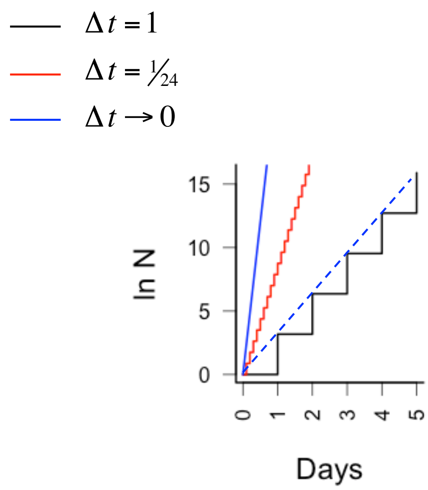


Differential equations (ODE's)

- Continuous treatment of individuals
- Continuous treatment of time
- Assumptions
  - large (infinite) populations
  - well-mixed contacts
  - homogenous individuals
  - exponential waiting times (memory-less)

Large population assumption

$$\frac{\beta SI}{N} \xrightarrow{\text{N large}} \beta$$

$$1$$

$$1/\gamma$$

$R_0 =$

Rate at which an infected individual produces new infections in a naïve population

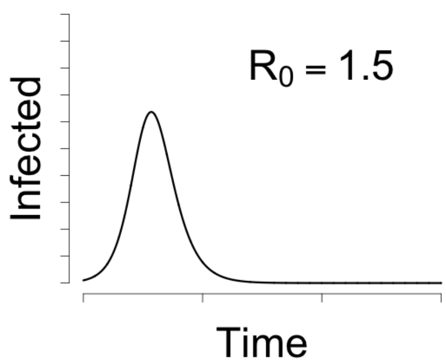
X

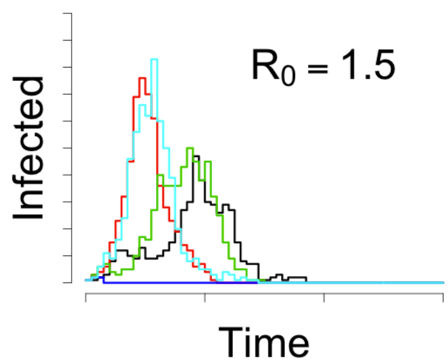
Proportion of new infections that become infectious

X

Average duration of infectiousness

Large population assumption





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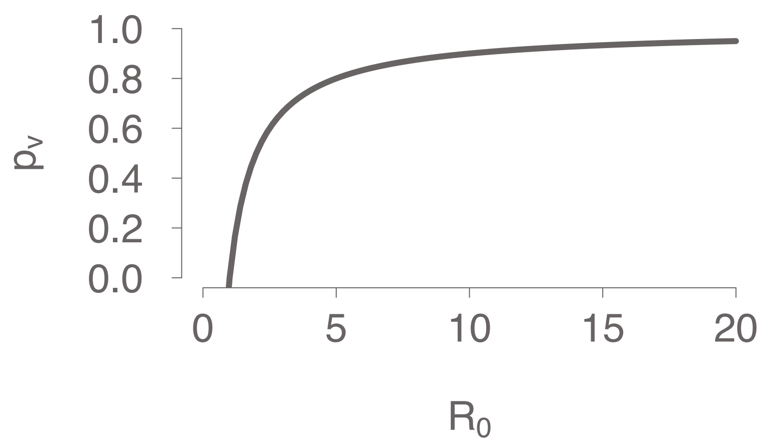
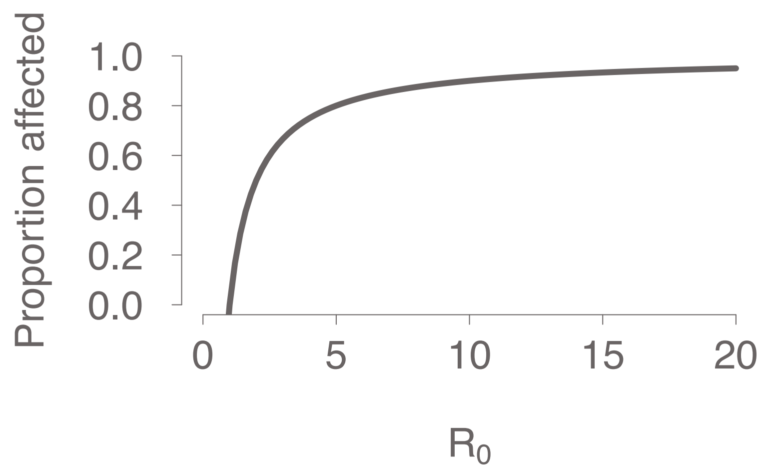
Homogeneity assumption



Homogeneity assumption



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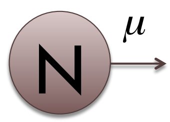
Exponential waiting times

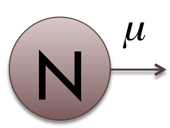
Exponential survival:



$$\frac{dN}{dt} = -\mu N$$

Exponential waiting times

Exponential survival:

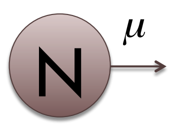


$$N_t = N_0 e^{-\mu t}$$

$$\frac{dN}{dt} = -\mu N$$

Exponential waiting times

Exponential survival:

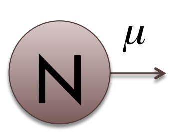


$$\frac{N_t}{N_0} = e^{-\mu t}$$

$$\frac{dN}{dt} = -\mu N$$

Exponential waiting times

Exponential survival:



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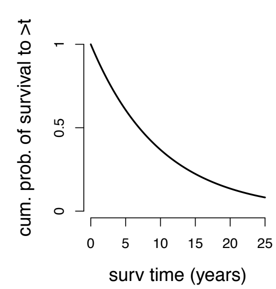
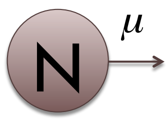
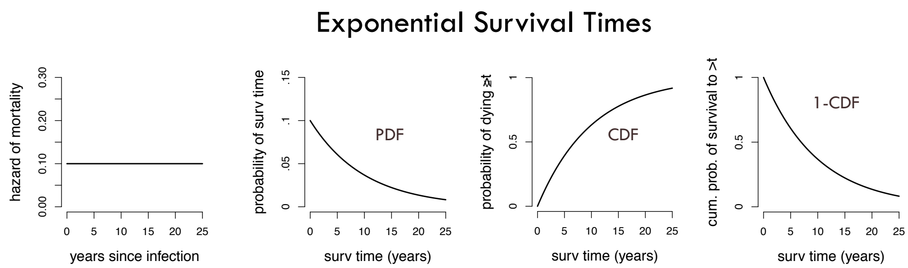


Exponential waiting times

Exponential survival:





hazard of mortality: deaths / (person*year)

probability of dying AT year t

probability of dying BY year t

probability of surviving PAST year t

Model taxonomy

	CONTINUOUS TREATMENT OF INDIVIDUALS (averages, proportions, or population densities)	DISCRETE TREATMENT OF INDIVIDUALS
DETERMINISTIC	CONTINUOUS TIME • Ordinary differential equations • Partial differential equations DISCRETE TIME • Difference equations (eg, Reed-Frost type models)	
STOCHASTIC	CONTINUOUS TIME • Stochastic differential equations DISCRETE TIME • Stochastic difference equations	CONTINUOUS TIME • Gillespie algorithm DISCRETE TIME • Chain binomial type models (eg, Stochastic Reed-Frost models)