Stochastic Models – Goals

- **Conceptual** sense of stochastic dynamics
- **Play** very mundanely with implementations
- Consider **regimes** of applicability / convenience

Stochastic Models – Outline

- Broad Considerations
- What is a probabilistic *process*?
- What is a *rate*?
- Basic Mathematics of *rates* and *flows*
- Different *implementations* of stochasticity

Real World and Model World

- **Real** World
  - Lots of unknowns
  - high complexity
- **Model** World
  - Everything is known
  - We control complexity
- *Model*
  - Mathematical representation of model world **Rules**
- *Scenario*
  - Mathematical representation of model world **Events**

Why Stochasticity?

- The Real World is stochastic
  - '*risks*' rather than clear '*causes*'
- Data would be stochastic even if the world were not
- Model world scenarios may not cluster close to average
  - Population / critical sub-populations not large enough?
  - '*rare*' events?
- Can be simpler than complicated flow models
  - Who contacts whom?
  - Complex variability of biology, environment

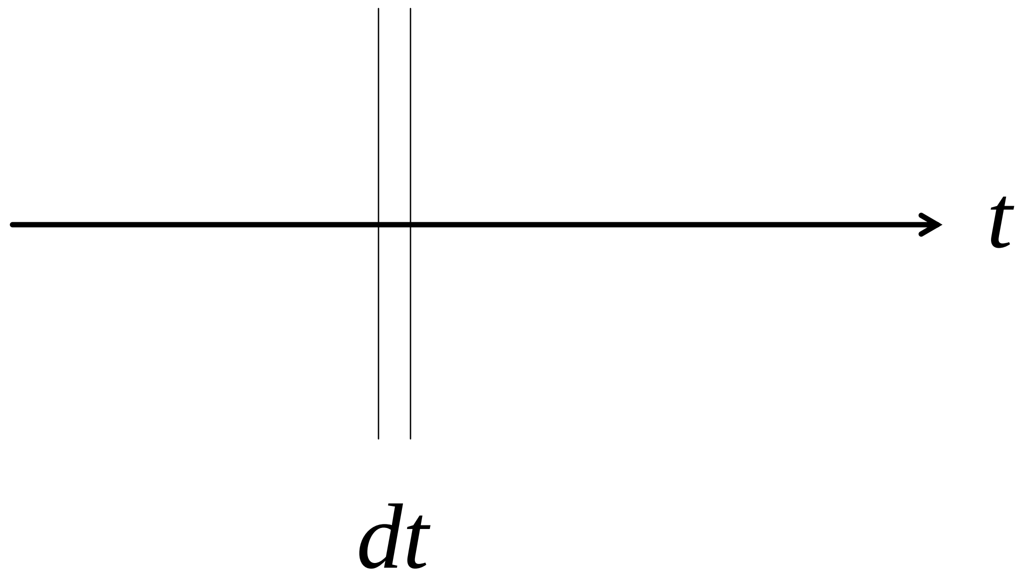
Choosing a Model

- Does it take in what we know and put out what we're curious about?
- Does it process the inputs according to our understanding
- Which aspects of `real world' have been left out / included?
- How do we evaluate the `severity' of the `imperfections'?

Implementing a Model

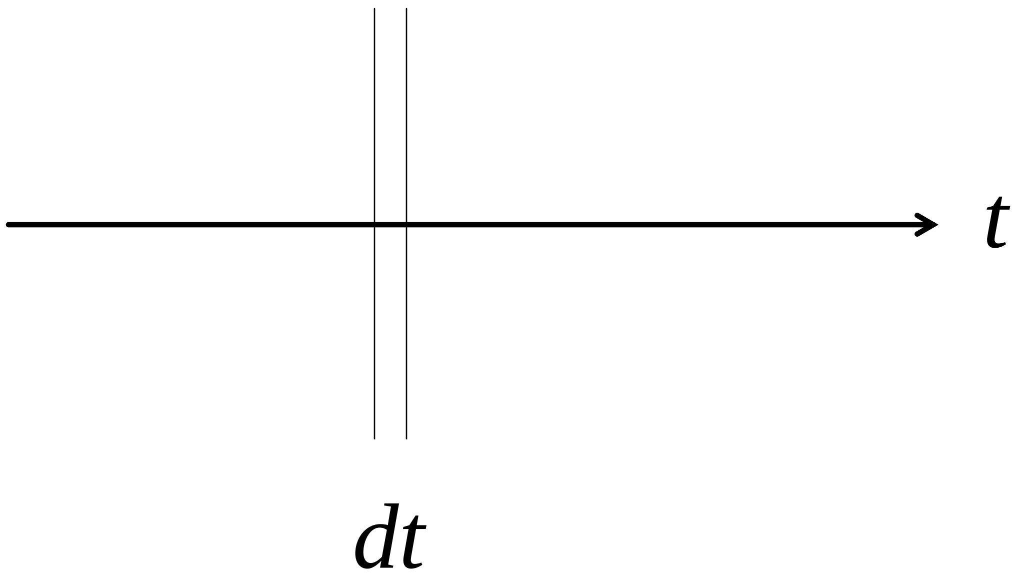
- Faithfulness of implementation
- Runtime
- Ease of (initial) deployment
- Ease of variation / maintenance

The Most Important Diagram



The Basic Dynamical Question

What (*tiny*) changes occur in the time step (*dt*)?



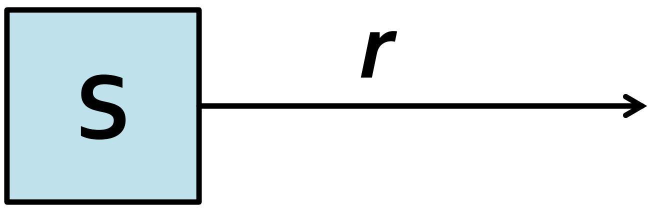
Basic Approaches

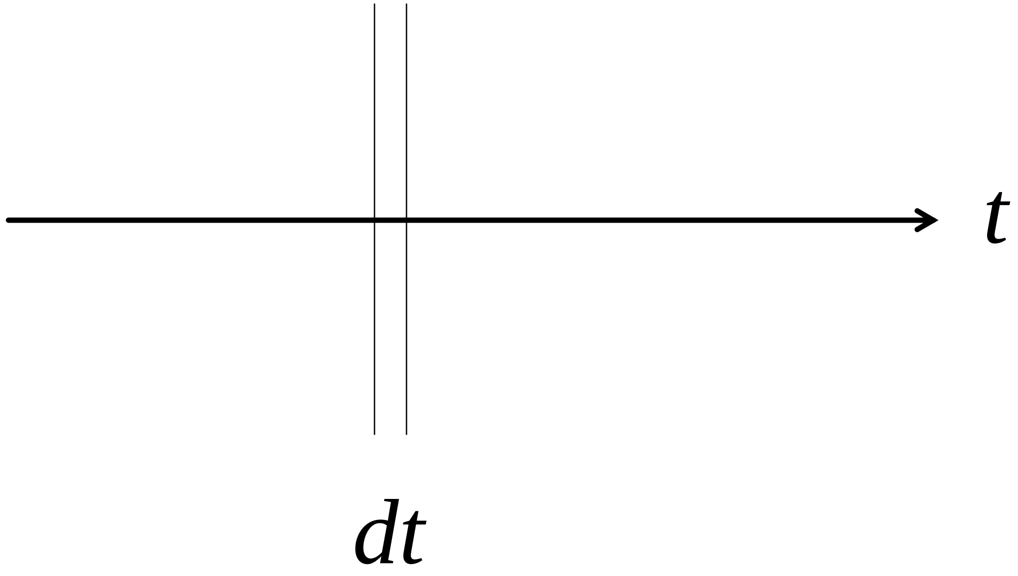
- Formal Solution
- Discretisation – naive flow in time steps
- Simulating ***events.***

  Can vary the approach:

  - for each *individual*, **vs** for *population* as a whole
  - As happening (or not) in time steps, *vs* generating times to next event.

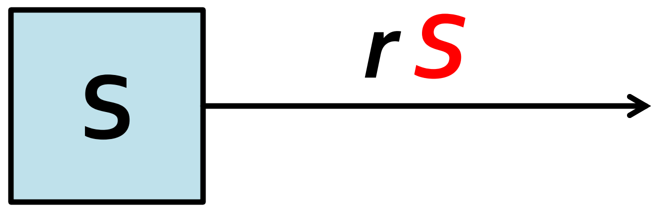
The Most Simple Model(?)

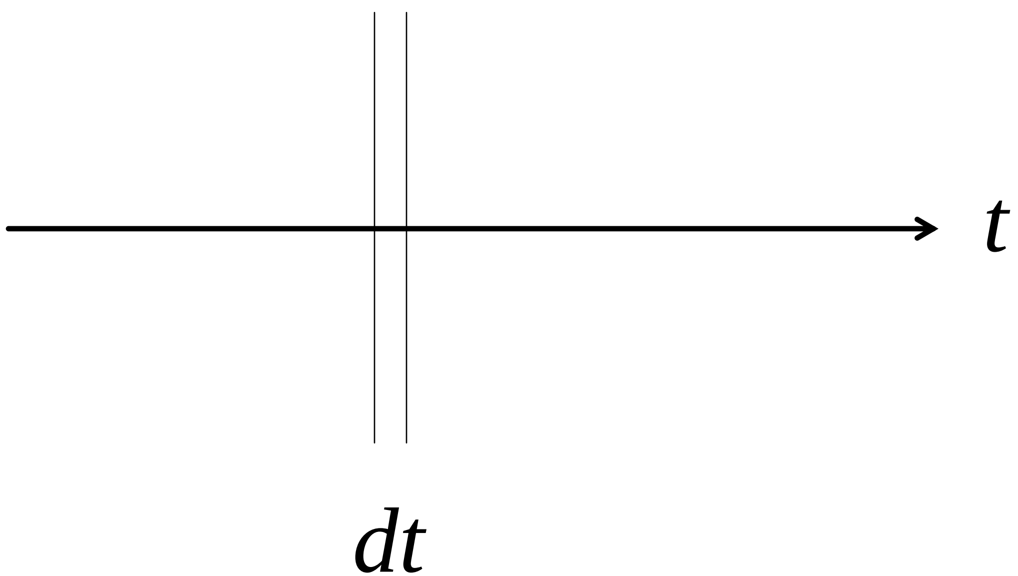




$$dS = (-)S(dP)$$
$$= -S(r\,dt)$$

The Most Simple Model(?)





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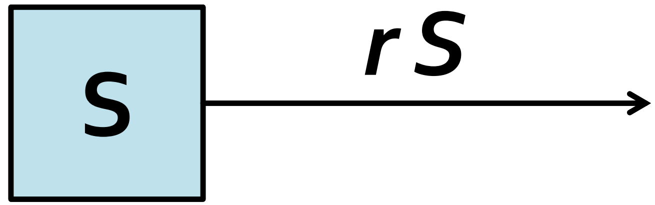
The Most Simple Model(?)



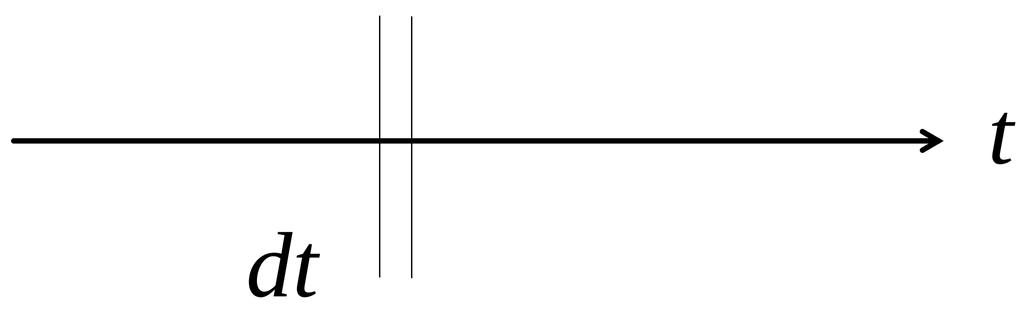
$$\frac{dS}{dt} = -rS$$

- How many people are susceptible at time = t?
- What is S(t) = ?

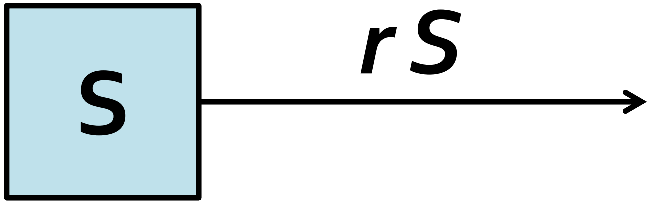
There are a number of approaches…

The Big Magic Trick

- While the (*tiny*) changes happen in (*dt*),
- The *rates of change* (the rules) stay the same.
- We just express the *rules* (in algebra).
- This leads to (differential) *equations.*
- We *solve* these using mature tools (*calculus*, *R*, *etc*).
- The solutions are *functions* of time, N(t).



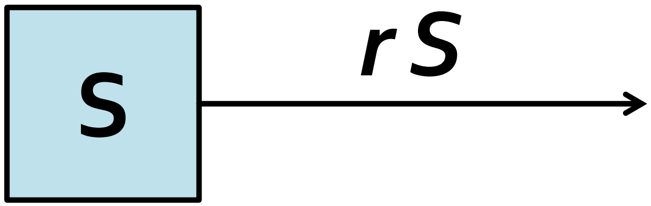
‘Exact’ Solution - Calculus



$$\left.\frac{dS}{dt} = -rS(t)\right\}$$ Differential equation

This one is not that difficult to solve

'Exact' Solution - Calculus



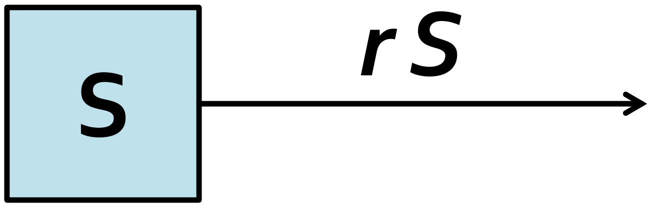
$$\frac{dS}{dt} = -rS(t)$$

Solving for S(t) yields:

$$S(t) = S_0 e^{-rt}$$

Analytic / Closed Form

Finite Time Steps

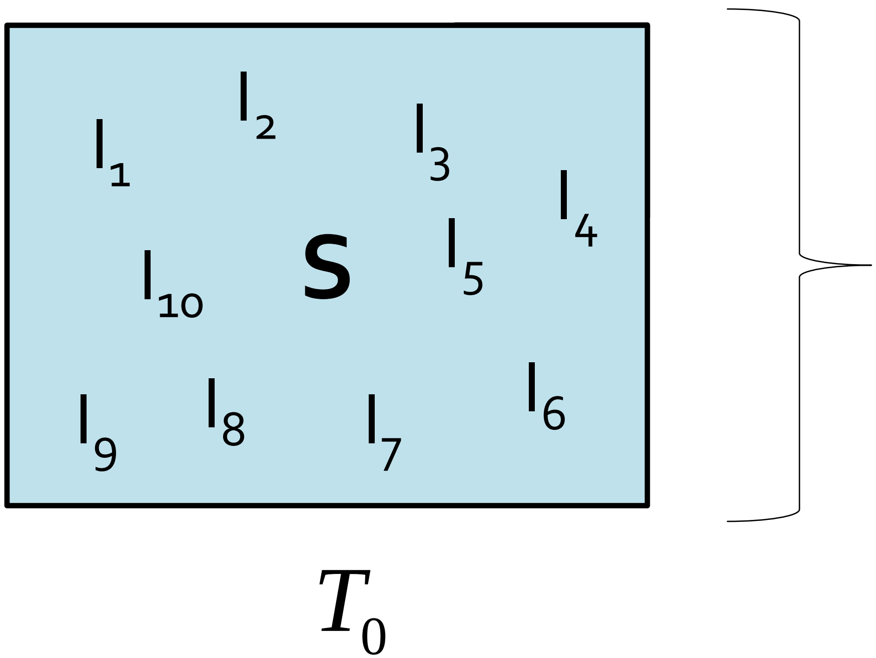


$$S(t+\Delta t) = S(t) - rS(t)\Delta t$$

Discrete / Numerical

‘Simulation’

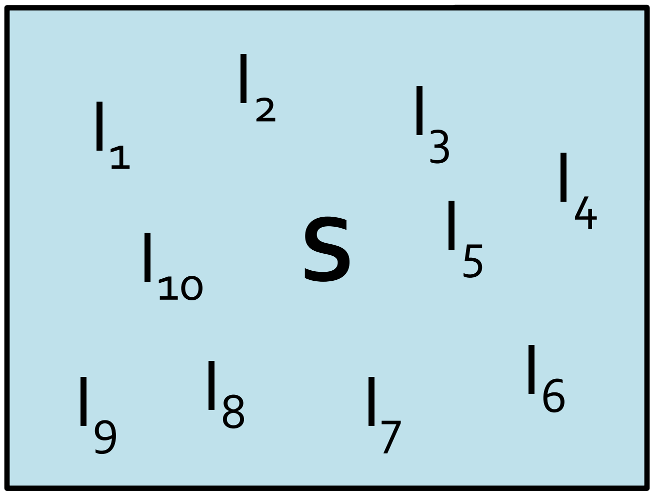
In each time step, each individual “leaves” S with probability = r



A population of 10 susceptible Individuals at T_0

'Simulation'

In each time step Δ, each individual "leaves" S with probability = δ



T_0

- Generate a random number, U, between 0 and 1 for each individual

- If $U > \delta$, individual stays in S

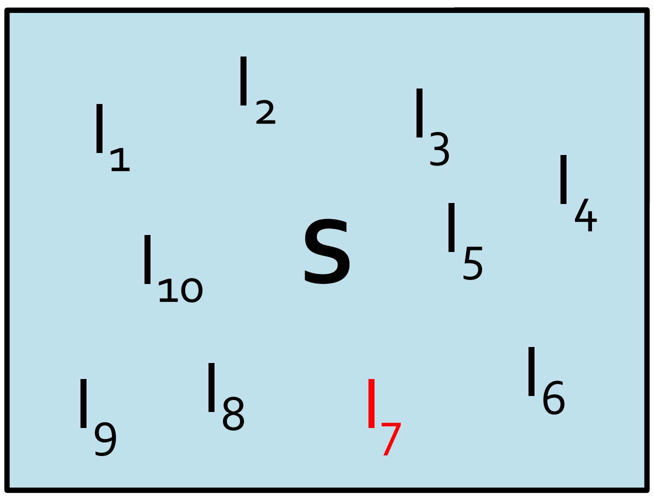
- If $U < \delta$, individual leaves S

- Suppose δ (=r Δ) = 0.05

U ~ Uniform on the interval [0, 1]

Random Events in a Time Step

In each time step Δ, each individual "leaves" S with probability = δ

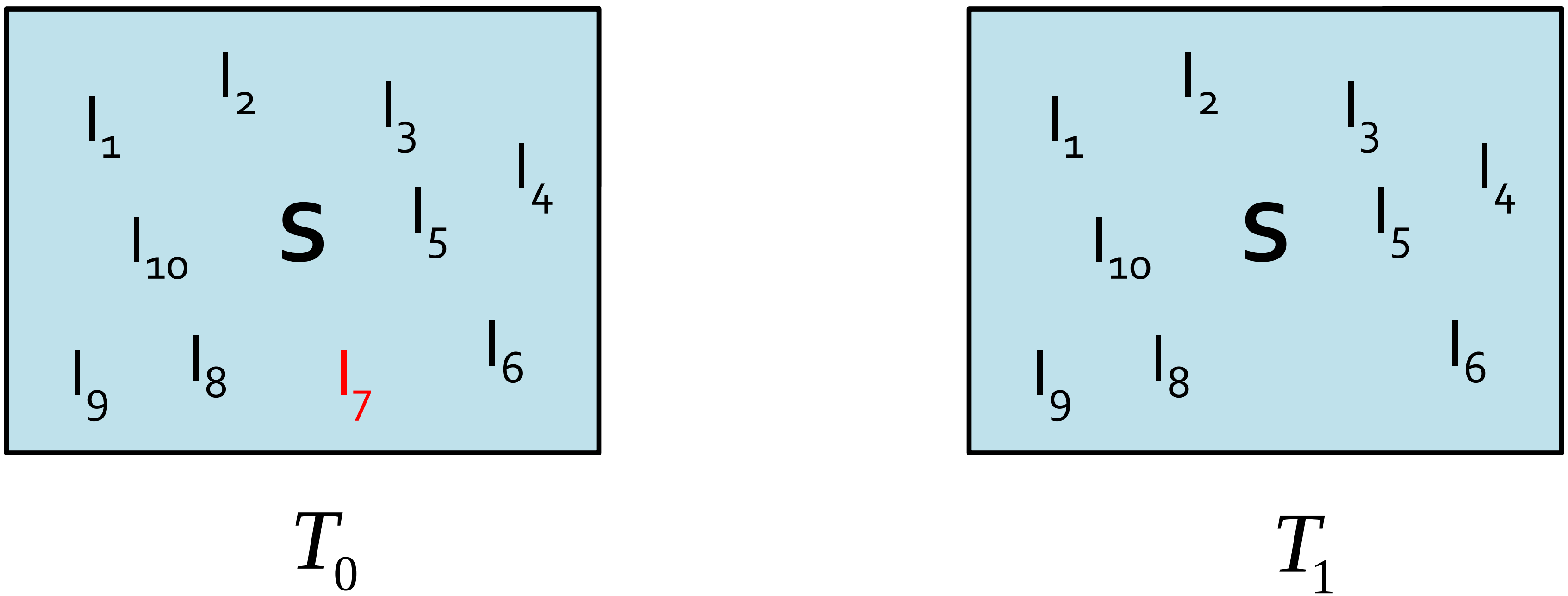


T_0

$Individual_i$	U_i
1	0.871
2	0.600
3	0.290
4	0.335
5	0.421
6	0.180
7	0.033
8	0.663
9	0.338
10	0.246

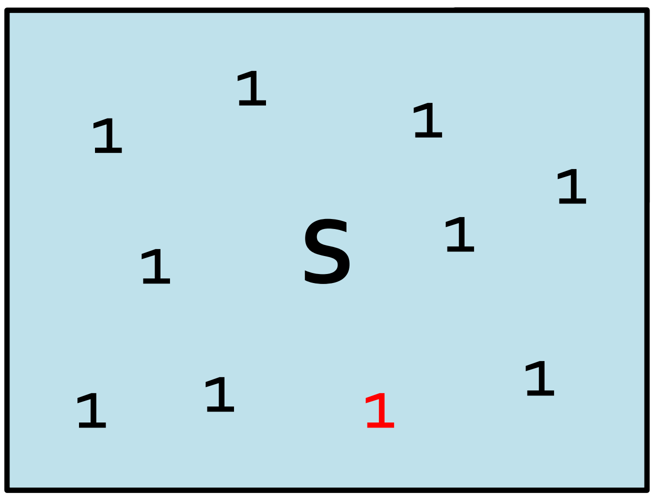
Completing the Time Step

In each time step Δ, each individual "leaves" S with probability = δ

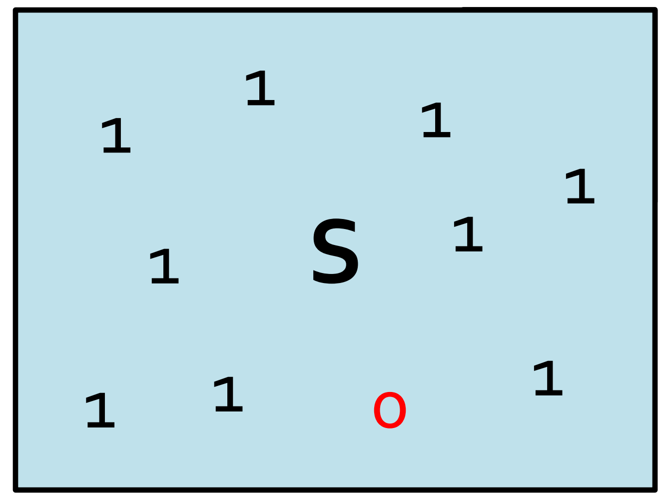


Abstracting Individuals

Using indicator variables to denote if an individual is still in "S"



T_0

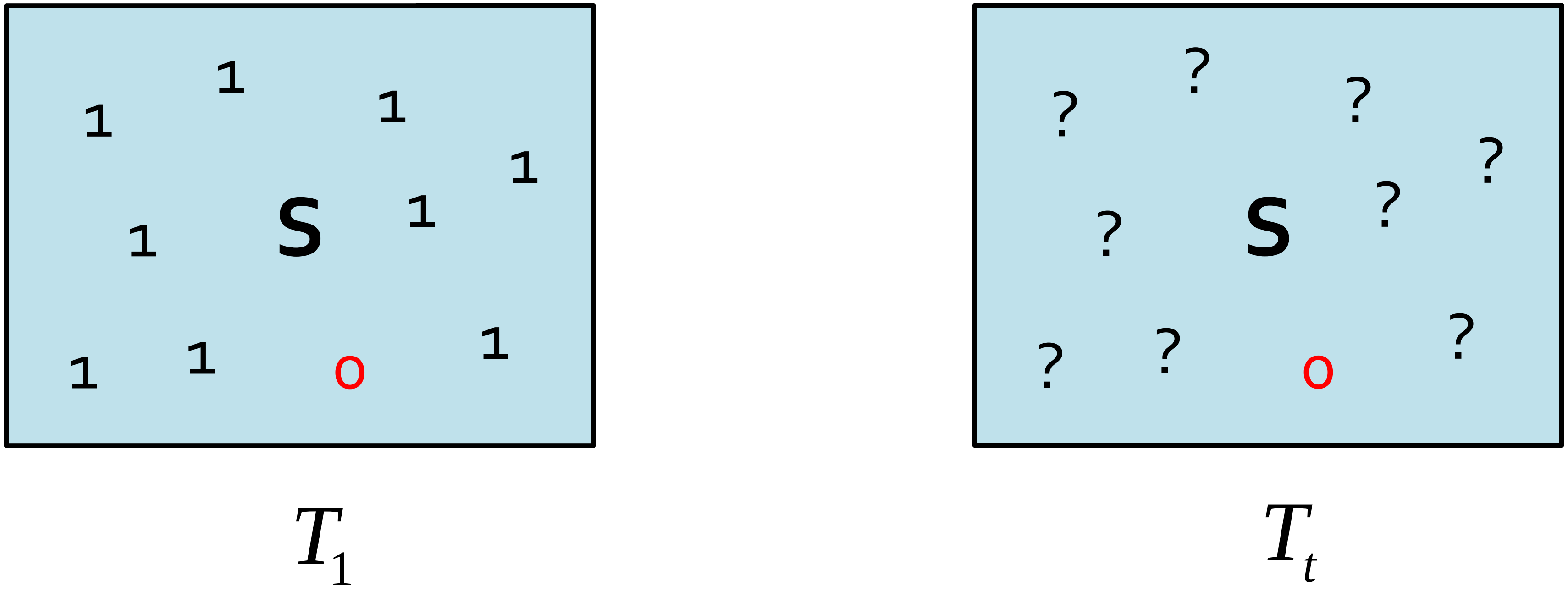


T_1

The Most Simple Model(?)

Approach #3

Repeat for the desired number of time steps to determine S(t)



Stochastic / Simulation

Discretisation ‘Error’

- In each time step Δ, each individual “leaves” S with probability = δ
- Can we really have $\delta = r\Delta$ when Δ is not small?
- ‘Discretisation error’
- How about $\delta = 1 - \exp(-r\Delta)$
- Taylor expansion: $f(x) = f(0) + f'(0)\, x^1 + f''(0)\, x^2 / 2! + \ldots$
- So then $\exp(x) = 1 - x + \ldots$
- And so $r\Delta \approx 1 - \exp(-r\Delta)$?

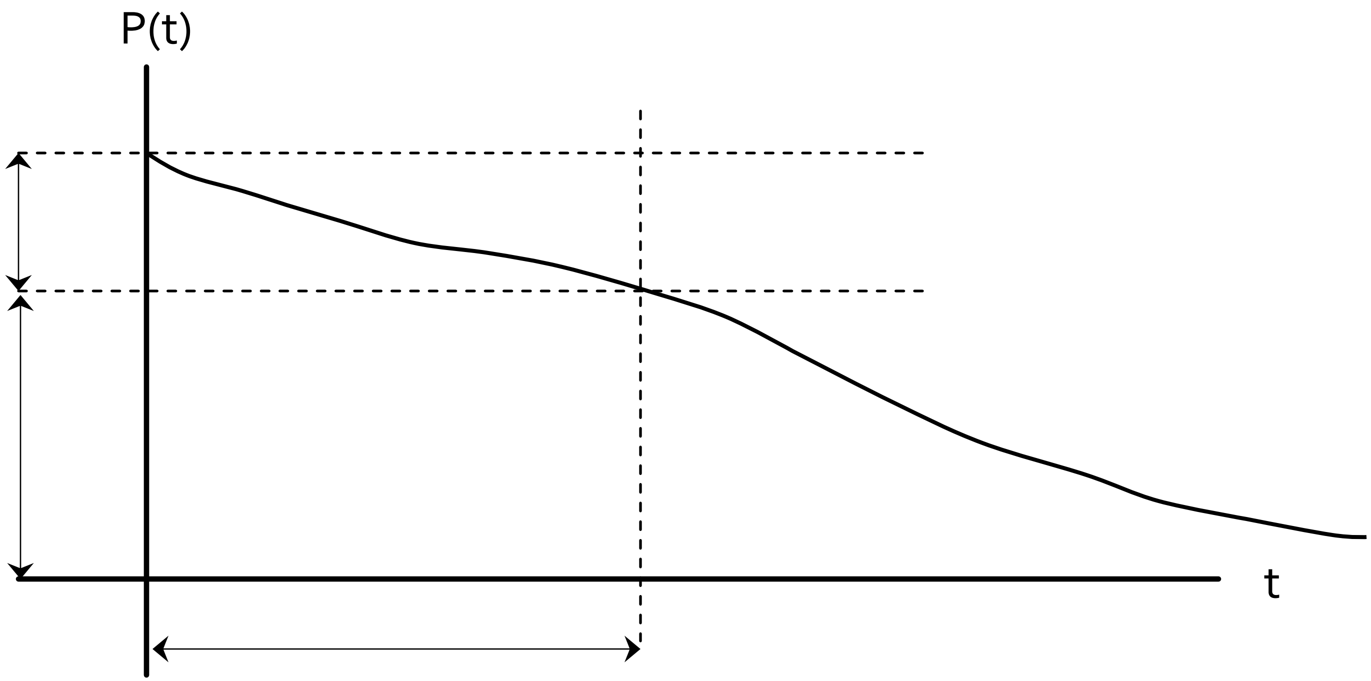
Events versus Time Steps

- In this simple model, we can discretise ‘correctly’
- This still only evaluates the system at discrete time points
- Why not calculate exact times at which population members leave?

$$S(t) = S_0 e^{-rt} \rightarrow \qquad P(t) = e^{-rt}$$

Scheduling Events

Transforming a uniform random number into a 'waiting time'



Scheduling Scheduling

- Should we schedule each exit up front?
- Or should we schedule the next exit
- then the next, …